QUANTUM YANG-BAXTER EQUATION, ARBITRARY GRADINGS AND EXCHANGE HECKE BRAIDINGS *†

W. Marcinek

Institute of Theoretical Physics, University of Wrocław,
Pl. Maxa Borna 9, 50-204 Wrocław, Poland

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Abstract

Generalization of the quantum Yang-Baxter equation solutions to an arbitrary grading is studied. The noncommutative differential calculi corresponding to such solutions is considered. The connection with the ordinary and supersymmetric solutions of the quantum Yang-Baxter equation is given.

KEYWORDS : Yang-Baxter operator, gradation, commutation factor, exchange operator, color R-matrix, deformation parameter, differential calculus

It is well known that the supersymmetry and the superspace formalism has been developed in the last 20 - 30 years in the context of Fermi-Bose correspondence, supergravity and string theory. A few generalizations of supersymmetry are known. The so-called color generalization of supersymmetry transformation has been introduced into physics by Rittenberg and Lukierski [1], Rittenberg and Wyler [2, 3]. The corresponding algebraic structures have been developed by several authors: Scheunert [4], Kobayashi and Nagamachi [5], Trostel [6], Kwaśniewski [7], Marcinek [8–11] and many others. Similar topics have been also considered by Matthes [12]. Next, the formal S-group of transformation as a generalization of supersymmetry, where S is a triangular Yang-Baxter operator (sometime called a symmetry), has been introduced by Gurevich [14–16]. The mathematical structures corresponding to such transformations has been studied by Lyubashenko [13], and further developed by others, see [11, 17, 18] for example. The generalization for arbitrary quasitriangular solution (braidings) of the quantum Yang-Baxter equation has been given by Majid [19–22]. In all of these above formalisms the tensor product of spaces is modified.

On the other hand the q-deformed structures and related quantum groups based on the usual (bosonic) tensor product, have been lately under intensive study in both, mathematics and theoretical physics. The corresponding formalism has been developed by several authors, Drinfeld, Jimbo, Woronowicz and many others. The well-known in theoretical physics quantum groups correspond to the q-deformation of universal enveloping algebras of simple Lie algebras. It is interesting that they admits the so-called universal R-matrix which also provide a solution of quantum Yang-Baxter equation. For example the R-matrix corresponding to quantum deformations of the groups of the series A_{n-1} satisfy the so-called Hecke condition. Note that we have in fact two attempts to quantum deformations. In the first approach only the tensor product is deformed, the algebra structure remains undeformed. In physical interpretation this case correspond to nonstandard statistics of particles. In the second approach the situation is opposite, the algebra structure is deformed, but the tensor product - not. In these two approach the concept of R-matrix play an important role. It is interesting that these two approach can be unified, see Majid [19] and Hlavaty [23]. In this way we obtain formalism with deformed tensor product and algebra structure. The corresponding formalism can be understood as a quantum deformations with braid group statistics. The quantum superspace studied by Ilinski and Uzdin [24] can be considered as an example of such structures. Graded structures studied previously by the author in [10] have been given as an another example for such structures.

In this paper we study some new graded structures which unify the color symmetry and quantum deformations. We introduce the concept of exchange braidings and color R-matrix. We study in details the exchange braiding and color R-matrix corresponding for quantum deformations with Hecke condition. A few examples for such R-matrices are given. They can be understood as a color generalizations of q-deformations corresponding to the Lie (super-) groups of the series A_{n-1}, [25,26]. The relation of our generalized graded structures with the standard (ungraded) or supersymmetric formalism is investigated. As a result we obtain that the odd (resp. even) color Hecke R-matrix can be reduced to a certain tensor product of Z_2-graded (resp. ungraded) solution of the quantum Yang-Baxter equation with Hecke condition and even color R-matrix. The generalization of Wess-Zumino [27] noncommutative differential calculi on a quantum plane is considered.

1 EXCHANGE BRAIDINGS

Let E be an arbitrary vector space over a field k. A linear operator $R : E \otimes E \longrightarrow E \otimes E$ such that

$$R_{12} \ R_{23} \ R_{12} = \ R_{23} \ R_{12} \ R_{23} \tag{1}$$

is said to be *a Yang-Baxter operator or a quasisymmetry or a braiding* on E, [16,17,28], where $R_{12} = R \otimes id$, and $R_{23} = id \otimes R$. If in addition

$$R^2 = id + (q - q^{-1})R, \tag{2}$$

then R is said to be a Hecke symmetry on E, or a Hecke braiding [16].

Now let E be a Γ-graded vector space over a field of complex numbers $k \equiv$ C, i.e. we have a direct sum decomposition $E = \bigoplus_{\alpha \in \Gamma} E_\alpha$. If the grading group Γ has a nontrivial subgroup Γ_0 of index 2 (i. e. the quotient group Γ/Γ_0 is isomorphic to the group Z_2), then the above gradation is said to be *odd*, in the opposite case it is called *even*. We denote by $\pi : \Gamma \longrightarrow \Gamma/\Gamma_0$ the corresponding quotient map. Assume that we have a homogeneous basis $\{e_i : i \in I\}$ of the space E, $I = \{1, 2, \ldots, N\}$, $N = \dim E$. This means that e_i is the i-th basis vector of E of grade $p(i)$, $p : I \longrightarrow \Gamma$ is a gradation mapping, i.e. $p(i) = \alpha$ if $e_i \in E_\alpha$. We introduce here a double index notation. We denote by $e_{i_\alpha,\alpha}$ the i_α-th basis vector of E. This means that for fixed α the set $\{e_{i_\alpha,\alpha} : i \in I_\alpha\}$ form a basis for E_α, where $I_\alpha := \{i \in I : p(i) = \alpha\}$. Observe that $\pi(\alpha) = +1$ or 0. The value of $\varepsilon_\alpha := (-1)^{\pi(\alpha)}$ is said to be *a parity* of α. Note that if $\varepsilon_\alpha = +1$ for every $\alpha \in G$, then the gradation is even. If there is an $\alpha \in \Gamma$ such that $\varepsilon_\alpha = -1$, then it is odd.

Let ε be a commutation factor on Γ, i.e. a mapping $\varepsilon : \Gamma \times \Gamma \longrightarrow C \setminus \{0\}$ such that $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1$ and $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$ for every $\alpha, \beta, \gamma \in \Gamma$ [4]. Note that $\varepsilon(\alpha, \alpha) = +1$, or -1 for every $\alpha \in \Gamma$. If $\varepsilon(\alpha, \alpha) = +1$ for every $\alpha \in \Gamma$, then ε is said to be even commutation factor. If there is an $\alpha \in \Gamma$ such that $\varepsilon(\alpha, \alpha) = -1$, then ε is said to be odd [5]. The set $\Gamma_0 = \{\alpha \in \Gamma : \varepsilon(\alpha, \alpha) = 1\}$ is obviously a subgroup of Γ of index at most 2. If the factor ε is odd, then the subgroup Γ_0 is nontrivial and the corresponding gradation is odd. Observe that here $\varepsilon_\alpha \equiv \varepsilon(\alpha, \alpha)$. For a given Γ-graded vector space E equipped with a homogeneous basis $\{e_{i_\alpha, \alpha} : i \in I_\alpha, \alpha \in \Gamma\}$ we introduce the concept of exchange operators.

A linear and homogeneous mapping $R : E \otimes E \longrightarrow E \otimes E$, where E is a Γ-graded vector space, which satisfies the Yang-Baxter equation (1) on E is said to be *a graded Yang-Baxter operator* on E. It is obvious that for every graded Yang-Baxter operator R we have the following decomposition

$$R := \bigoplus_{\alpha, \beta \in \Gamma} R(\alpha, \beta), \tag{3}$$

where $R(\alpha, \beta) : E_\alpha \otimes E_\beta \longrightarrow E_\beta \otimes E_\alpha$ is the homogeneous component of R with respect to the $\Gamma \times \Gamma$-gradation of the tensor product $E \otimes E$. Every such component $R(\alpha, \beta)$ of R is said to be *a color exchange operator* or (*a color R-matrix*) on E. The following result is obvious: the quantum Yang-Baxter equation (1) for a graded Yang-Baxter operator R on E is equivalent to the following family of equations

$$R_{12}(\beta, \gamma)R_{23}(\alpha, \gamma)R_{12}(\alpha, \beta) = R_{23}(\alpha, \beta)R_{12}(\alpha, \gamma)R_{23}(\beta, \gamma), \tag{4}$$

defined for the family of exchange operators $\{R(\alpha, \beta) : \alpha, \beta \in \Gamma\}$, on $E_\alpha \otimes E_\beta \otimes E_\gamma$ for every $\alpha, \beta, \gamma \in \Gamma$. The family of relations (4) is called *a color exchange quantum Yang-Baxter equation*. The above result means that every graded Yang-Baxter operator R on E can be determined by a family of color exchange operators $\{R(\alpha, \beta) : \alpha, \beta \in \Gamma\}$ defined for a family of spaces $\{E_\alpha : \alpha \in \Gamma\}$, where E_α is the α-th component of E with respect to a given Γ-gradation. The family $\{R(\alpha, \beta)\}$ is sometime said to be *a color exchange* braidings (or symmetries) for $\{E_\alpha : \alpha \in \Gamma\}$. Let us consider an example: a family of operators $\{(R_\varepsilon)(\alpha, \beta) : \alpha, \beta \in \Gamma\}$ given by the following relation

$$R_\varepsilon(\alpha, \beta)(e_{i_\alpha, \alpha} \otimes e_{j_\beta, \beta}) := \varepsilon(\alpha, \beta) \, e_{j_\beta, \beta} \otimes e_{i_\alpha, \alpha}, \tag{5}$$

where ε is a commutation factor on the grading group Γ, is a family of exchange operators for $\{E_\alpha : \alpha \in \Gamma\}$. This family define a graded Yang-Baxter operator R_ε on E. The operator R_ε is said to be color.

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If the commutation factor ε on Γ is even (resp. odd), then these above color operators or symmetries are said to be even or even (resp. odd). The above graded Yang-Baxter operator is diagonal with respect to both classes of indices: external and internal. Now let us consider the case in which a given graded Yang-Baxter operator is diagonal only in external (grading) indices. To this goal we introduce some special grading of E_Γ. Let E_Γ be a Γ-graded space such that every component E_α of E with respect to a given Γ-gradation has the same dimension equal to k, i.e. $\dim(E_\alpha) = k$ for every $\alpha \in \Gamma$. This means that all these components E_α of E_Γ are isomorphic one to other. In other words, there exist a space E such that each E_α is isomorphic to E. In this way every E_α can be understand as a copy of the space E and the Γ-graded vector space E_Γ which is direct sum of all spaces E_α can be identified with the product $E \times \Gamma$. Hence we have here an analogy with the topological concept of covering spaces in a certain algebraic sense. The space E_Γ play in fact a role of algebraic covering of the "base" space E with a discrete "fiber" Γ. Let us denote by e_i the i-th basis vector of E, $i = 1, \ldots, k$. The one-to one correspondence between spaces E_Γ and $E \times \Gamma$ is given by $e_{i_\alpha,\alpha} \leftrightarrow (e_i, \alpha)$, where $i_\alpha = 1, \ldots, N$, $i = 1, \ldots, k$, $\alpha \in \Gamma$. The tensor product is $E_\Gamma \otimes E_\Gamma$.

Observe that for every element $e_{i_\alpha,\alpha}$ of E_α there is a corresponding element e_i of E. We define a set of components $R(\alpha, \beta)$ for an operator R on a Γ-graded space E by the following formula

$$R(\alpha, \beta) = q^{1-2\pi(\alpha)}\, \varepsilon(\alpha, \alpha)\, \Sigma_{i \in I_\alpha}\, e_{i,\alpha\ i,\alpha} \otimes e_{i,\alpha\ i,\alpha}$$
$$+\Sigma_{i \in I_\alpha, j \in I_\beta, i \neq j}\, \varepsilon(\alpha, \beta)\, e_{i,\alpha\ j,\beta} \otimes e_{j,\beta\ i,\alpha} + \varepsilon(\alpha, \alpha)(q - q^{-1})$$
$$\times \Sigma_{i \in I_\alpha, j \in I_\beta, i < j}\, e_{i,\alpha\ i,\alpha} \otimes e_{j,\beta\ j,\beta}, \tag{6}$$

where ε is a commutation factor on Γ, q is a deformation parameter, $q \in C \setminus \{0\}$, and $(e_{i,\alpha\ j,\beta})_{k,\gamma\ l,\delta} = \delta_{i,\alpha\ k,\delta}\delta_{j,\beta\ l,\alpha}$.

Note that the corresponding family of mappings $R(\alpha, \beta) : E_\alpha \otimes E_\beta \longrightarrow E_\beta \otimes E_\alpha$ is given explicit by the equation

$$R(\alpha, \beta)(e_{i,\alpha} \otimes e_{j,\beta}) = \begin{cases} q^{1-2\pi(\alpha)}\varepsilon(\alpha, \alpha)e_{i,\alpha} \otimes e_{i,\alpha}, & \text{for } i = j, \\ \varepsilon(\alpha, \alpha)(q - q^{-1})e_{i,\beta} \otimes e_{j,\alpha} + \varepsilon(\alpha, \beta)e_{j,\beta} \otimes e_{i,\alpha}, & \text{for } i < j, \\ \varepsilon(\alpha, \beta)e_{j,\beta} \otimes e_{i,\alpha}, & \text{for } i > j. \end{cases} \tag{7}$$

If the commutation factor ε on Γ is even, then the family (10) is said to be *even color Hecke braidings*. If ε is odd, then this family is also said to be *odd*. Next we use the notation: $\varepsilon_\alpha := \varepsilon(\alpha, \beta) \equiv (-1)^{\pi(\alpha)}$.

For the family of operators $\{R(\alpha, \beta)\}$ given by the equation (10) we have **(i)**

the Yang-Baxter equation (4), and **(ii)** the corresponding Hecke condition

$$R^2(\alpha, \beta) = id + (-1)^{\pi(\alpha)}(q - q^{-1})R(\alpha, \beta). \qquad (8)$$

To show this first we prove the Yang-Baxter equation (4) for the operation (10). For $i < j < k$ we calculate

$$R_{12}(\alpha, \beta)(e_{i,\alpha} \otimes e_{j,\beta} \otimes x_{k,\gamma}) = R(\alpha, \beta)(e_{i,\alpha} \otimes e_{j,\beta}) \otimes e_{j,\gamma}$$
$$= \varepsilon_{i,\alpha}(q - q^{-1})e_{i,\alpha} \otimes e_{j,\beta} \otimes e_{j,\gamma} + \varepsilon(\alpha, \beta)e_{j,\beta} \otimes e_{i,\alpha} \otimes e_{j,\gamma}, \qquad (9)$$

and

$$R_{23}(\alpha, \gamma)R_{12}(\alpha, \beta)(e_{i,\alpha} \otimes e_{j,\beta} \otimes e_{j,\gamma}) = \varepsilon_{i,\alpha}(q - q^{-1})e_{i,\alpha} \otimes e_{j,\beta} \otimes e_{j,\gamma}$$
$$+\varepsilon_{i,\alpha}(q - q^{-1})\varepsilon(\beta, \gamma)e_{i,\alpha} \otimes e_{j,\gamma} \otimes e_{j,\beta} + \varepsilon_{i,\alpha}(q - q^{-1})\varepsilon(\alpha, \beta)e_{j,\beta} \otimes e_{i,\alpha} \otimes e_{j,\gamma}$$
$$+\varepsilon(\alpha, \beta + \gamma)e_{j,\beta} \otimes e_{j,\gamma} \otimes e_{i,\alpha}.$$

Hence for the left hand side of the Yang-Baxter equation (4) we obtain

$$R_{12}R_{23}R_{12}(e_{i,\alpha} \otimes e_{j,\beta} \otimes e_{j,\gamma}) = [\varepsilon_{i,\alpha}(q - q^{-1})^3 + \varepsilon_{i,\alpha}(q - q^{-1})]e_{i,\alpha} \otimes e_{j,\beta} \otimes e_{j,\gamma}$$
$$+(q - q^{-1})^2\varepsilon(\alpha, \beta)e_{j,\beta} \otimes e_{i,\alpha} \otimes e_{j,\gamma} + (q - q^{-1})^2\varepsilon(\beta, \gamma)e_{i,\alpha} \otimes e_{j,\gamma} \otimes e_{j,\beta}$$
$$+\varepsilon_{i,\alpha}(q - q^{-1})\varepsilon(\alpha + \beta, \gamma)e_{j,\gamma} \otimes e_{i,\alpha} \otimes e_{j,\beta} + \varepsilon_{i,\alpha}(q - q^{-1})\varepsilon(\alpha, \beta + \gamma)e_{j,\beta} \otimes e_{j,\gamma} \otimes e_{i,\alpha}$$
$$+\varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)e_{j,\gamma} \otimes e_{j,\beta} \otimes e_{i,\alpha},$$

and for the right-hand side

$$R_{23}R_{12}R_{23}(e_{i,\alpha} \otimes e_{j,\beta} \otimes e_{j,\gamma}) = [\varepsilon_{i,\alpha}(q - q^{-1})^3 + \varepsilon_{i,\alpha}(q - q^{-1})]e_{i,\alpha} \otimes e_{j,\beta} \otimes e_{j,\gamma}$$
$$+(q - q^{-1})^2\varepsilon(\beta, \gamma)e_{i,\alpha} \otimes e_{j,\gamma} \otimes e_{j,\beta} + (q - q^{-1})^2\varepsilon(\alpha, \beta)e_{j,\beta} \otimes e_{i,\alpha} \otimes e_{j,\gamma}$$
$$+\varepsilon_{i,\alpha}(q - q^{-1})\varepsilon(\alpha, \beta + \gamma)e_{j,\beta} \otimes e_{j,\gamma} \otimes e_{i,\alpha} + \varepsilon_{i,\alpha}(q - q^{-1})\varepsilon(\alpha + \beta, \gamma)e_{j,\gamma} \otimes e_{i,\alpha} \otimes e_{j,\beta}$$
$$+\varepsilon(\alpha, \beta)\varepsilon(\beta, \gamma)\varepsilon(\alpha, \gamma)e_{j,\gamma} \otimes e_{j,\beta} \otimes e_{i,\alpha}.$$

The proof in the remaining cases is obvious. We have for $i < j$

$$R^2(\alpha, \beta)(e_{i,\alpha} \otimes e_{j,\beta}) = R(\beta, \alpha)((q - q^{-1})e_{i,\alpha} \otimes e_{j,\beta} + \varepsilon(\alpha, \beta)e_{j,\beta} \otimes e_{i,\alpha})$$
$$= \varepsilon_{i,\alpha}(q - q^{-1})^2 x_{i,\alpha} \otimes x_{j,\beta} + \varepsilon_{i,\alpha}(q - q^{-1})\varepsilon(\alpha, \beta)e_{j,\beta} \otimes e_{i,\alpha} + e_{i,\alpha} \otimes e_{j,\beta}$$
$$= (id + (q - q^{-1})R)(e_{i,\alpha} \otimes e_{j,\beta}).$$

The operation defined by the equation (10) is called *a color Hecke R-matrix* and the corresponding family of exchange mappings is said to be *a color Hecke braiding* on a Γ-graded space E corresponding to the commutation factor ε.

Let us consider some particular examples. We define a operator R on a Γ-graded space E by the following formula

$$R = \bigoplus_{\alpha,\beta\in\Gamma} (q^{1-2\pi(\alpha)} \, \varepsilon(\alpha,\alpha) \, \Sigma_{i\in I_\alpha} \, e_{i,\alpha\ i,\alpha} \otimes e_{i,\alpha\ i,\alpha}$$

$$+ \Sigma_{i\in I_\alpha,j\in I_\beta,i\neq j} \, \varepsilon(\alpha,\beta) \, e_{j,\beta\ i,\alpha} \otimes e_{i,\alpha\ j,\beta} + \varepsilon(\alpha,\alpha)(q-q^{-1})$$

$$\times \Sigma_{i\in I_\alpha,j\in I_\beta,i<j} \, e_{i,\alpha\ i,\alpha} \otimes e_{j,\beta\ j,\beta}), \tag{10}$$

where ε is a commutation factor on Γ, q is a deformation parameter, $q \in C\backslash\{0\}$, and $(e_{i,\alpha\ j,\beta})_{k,\gamma\ l,\delta} = \delta_{i,\alpha\ k,\delta}\delta_{j,\beta\ l,\alpha}$. Note that the corresponding family of mappings $R(\alpha,\beta) : E_\alpha \otimes E_\beta \longrightarrow E_\beta \otimes E_\alpha$ is given explicit by the equation

$$R(\alpha,\beta)(e_{i,\alpha}\otimes e_{j,\beta}) = \begin{cases} q^{1-2\pi(\alpha)}\varepsilon(\alpha,\alpha)e_{i,\alpha} \otimes e_{i,\alpha}, & \text{for } i=j, \\ \varepsilon(\alpha,\alpha)(q-q^{-1})e_{i,\beta} \otimes e_{j,\alpha} + \varepsilon(\alpha,\beta)e_{j,\beta} \otimes e_{i,\alpha}, & \text{for } i<j, \\ \varepsilon(\alpha,\beta)e_{j,\beta} \otimes e_{i,\alpha}, & \text{for } i>j. \end{cases} \tag{11}$$

Let us consider a Γ-graded vector space E equipped with a homogeneous basis $e_i : i \in I$ and a grading map $p : I \longrightarrow \Gamma$. Let us assume for simplicity that $\dim(E_\alpha) = 2$ for every every $\alpha \in \Gamma$ and $\varepsilon_i = 1$ for $i = 1, 2$. We use the notation $e_{1,\alpha} := x_\alpha$, $e_{2,\beta} := y_\beta$.

In this case the color Hecke R-matrix (10) becomes

$$R(\alpha,\beta) = \tag{12}$$

$$\begin{bmatrix} q^{1-2\pi(\alpha)}\varepsilon(\alpha,\beta)\delta_\alpha^\delta\delta_\beta^\gamma\delta_{\alpha\beta} & 0 & 0 & 0 \\ 0 & (q-q^{-1})\delta_\alpha^\gamma\delta_\beta^\delta & \varepsilon(\alpha,\beta)\delta_\alpha^\delta\delta_\beta^\gamma & 0 \\ 0 & \varepsilon(\beta,\alpha)\delta_\alpha^\delta\delta_\beta^\gamma & 0 & 0 \\ 0 & 0 & 0 & q^{1-2\pi(\alpha)}\varepsilon(\alpha,\beta)\delta_\alpha^\delta\delta_\beta^\gamma\delta_{\alpha\beta} \end{bmatrix},$$

where $\alpha, \beta, \gamma, \delta \in \Gamma$. The corresponding exchange braiding has the following form

$$\begin{aligned} R(x_\alpha \otimes x_\alpha) &= q^{1-2\pi(\alpha)}\varepsilon(\alpha,\alpha)x_\alpha \otimes x_\alpha, \\ R(x_\alpha \otimes y_\beta) &= (q-q^{-1})x_\alpha \otimes y_\beta + \varepsilon(\alpha,\beta)y_\beta \otimes x_\alpha, \\ R(y_\beta \otimes x_\alpha) &= \varepsilon(\beta,\alpha)x_\alpha \otimes y_\beta, \\ R(y_\beta \otimes y_\beta) &= q^{1-2\pi(\beta)}\varepsilon(\beta,\beta)y_\beta \otimes y_\beta. \end{aligned} \tag{13}$$

Let $I = \{1, 2, 3, 4\}$, $\Gamma = Z_2$, and the grading map is given by: $p(1) = p(2) = 0, p(3) = p(4) = 1$. We have the following supercommutation factor on Z_2

$$\varepsilon(\alpha,\beta) = (-1)^{\alpha\beta} \tag{14}$$

for $\alpha, \beta \in Z_2$. Here the color R-matrix (10) takes the following form

$$R = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & (q - q^{-1}) & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{bmatrix}.$$

This is the well-known (braid) R-matrix corresponding to quantum group $gl(1|1)$, see e.g. [30].

Let E be a Γ-graded vector space, where Γ be an Abelian group generated by ξ_i, $i = 1, ..., k$, and we have $\dim E_\alpha = 1$ for every $\alpha \in \Gamma$. We assume here that the grading map $p : I \longrightarrow \Gamma$ is given by

$$p(i) = \xi_i. \tag{15}$$

This is the so-called standard gradation [11]. We use the notation $\varepsilon(\xi_i, \xi_j) = \varepsilon_{ij}$. Let $\Gamma = Z \oplus ... \oplus Z$ (k-summands). Here we have [11, 29]

$$\varepsilon_{ij} = (-1)^{s_{ij}} z^{a_{ij}}, \tag{16}$$

where $s_{ij} = s_{ji}$ and $a_{ij} = -a_{ji}$ are integer valued matrices, $z \in C - \{0\}$ is an another deformation parameter, and $\xi_i = (0...1...0)$, (1 on the i-th place). Here the subgroup Γ_0 is generated by the set $\{\xi_i : s_{ii} = 0\}$. In this case the color Hecke R-matrix (10) can be given in the following form

$$R(\xi_i, \xi_j) = q^{1-2s_{ii}}(-1)^{s_{ii}}\Sigma_{i \in I}e_{ii} \otimes e_{ii} + \Sigma_{i,j \in I, i \neq j}\varepsilon_{ij}e_{ij} \otimes e_{ji} + (q + q^{-1})\Sigma_{i,j \in I, i < j}e_{ii} \otimes e_{jj}, \tag{17}$$

here ε_{ij} is given by the equation (16). Note that in the case of $s_{ii} = 0$ for all $i \in I$, the the above R-matrix is similar the so-called multiparameter R-matrix, see [32, 33]. The multiparametric R-matrices are in fact the color Hecke R-matrices with Z^k -gradings defined by (15) and (16), where $s_{ii} = 0$ for every $i \in I$.

If $\Gamma = Z_N \oplus ... \oplus Z_N$ (k-summands), $N > 2$, and

$$\varepsilon_{ij} = exp((2\pi i/N)\Omega_{ij}), \tag{18}$$

where $\Omega_{ij} = -\Omega_{ji}$ is an integer-valued matrix, then the color Hecke R-matrix (10) takes the following form

$$R(\xi_i, \xi_j) = q\Sigma_{i \in e}e_{ii} \otimes e_{ii} + \Sigma_{i,j \in I, i \neq j}exp((2\pi i/N)\Omega_{ij})e_{ji} \otimes e_{ij} + (q + q^{-1})\Sigma_{i,j \in I, i < j}e_{ii} \otimes e_{jj}. \tag{19}$$

The above R-matrix can be understood as the R-matrix corresponding to the deformation of the anyonic statistics [34].

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2 REDUCTION OF COLOR-HECKE BRAIDING

Let E be a Γ-graded vector space equipped with a homogeneous basis $\{e_{i,\alpha} : i \in I_\alpha; \alpha \in \Gamma\}$. We assume here that an odd color Hecke R-matrix on E corresponding to a commutation factor ε on Γ is given. It follows immediately from our definitions that the subgroup Γ_0 of Γ is nontrivial, i.e. the quotient Γ/Γ_0 is isomorphic to the Z_2 group. Let us denote by $\tilde{E}$ the same vector space E but equipped with Z_2 gradation defined by the following formulae

$$\tilde{E} = \tilde{E}_0 \oplus \tilde{E}_1, \quad \tilde{E}_0 = \bigoplus_{\alpha \in \Gamma_0} E_\alpha, \quad \tilde{E}_1 = \bigoplus_{\alpha \in \Gamma_1} E_\alpha, \tag{20}$$

where $\Gamma_1 := \{\alpha \in \Gamma : \varepsilon(\alpha, \alpha) = -1\}$. The corresponding Z_2-grading mapping $\delta : I \longrightarrow Z_2$ is given here by the following formula

$$\delta = \pi \circ p. \tag{21}$$

We denote by $\tilde{e}_{i,\pi(\alpha)}$ the same element $e_{i,\alpha}$ of E but equipped with the above Z_2-gradation instead of the initial Γ-gradation. We introduce a Γ-graded vector space E' such that

$$E' = \bigoplus_{\alpha \in \Gamma} E'_\alpha, \quad \dim E'_\alpha = 1 \quad for \quad every \quad \alpha \in \Gamma, \tag{22}$$

and E' is spanned by element ρ_α, $\alpha \in \Gamma$. We denote by $\bar{E}$ the subspace of the tensor product $\tilde{E} \otimes E'$ spanned by elements of the form $\tilde{e}_{i,\alpha} \otimes \rho_\alpha$. Observe that the space $\bar{E}$ is a Γ-graded vector space isomorphic to E. The space $\bar{E}$ is also Z_2-graded. The Z_2-gradation is given by the grading map (21). It is known that the commutation factor ε on Γ can be given in the following form

$$\varepsilon(\alpha, \beta) = (-1)^{\pi(\alpha)\pi(\beta)}\delta(\alpha, \beta), \tag{23}$$

where $\alpha, \beta \in \Gamma$, and $\delta(\alpha, \beta) := \sigma(\alpha, \beta)/\sigma(\beta, \alpha)$, and σ is a certain 2-cocycle on Γ, see [4, 8] for example.

We define a linear mapping $s : E \longrightarrow \bar{E}$ called *a superization* by the relation $s(e_{i,\alpha}) := \tilde{e}_{i,\pi(\alpha)} \otimes \rho_\alpha$. Note that the superization mapping s is not unique, here some additional degree of freedom appears, see [7, 35] for more details concerning with the homological description of the superization procedure. The tensor product of spaces with Z_2 gradation is denoted by $\otimes'$. We have here the following relation between our tensor products $(s \otimes' s)(e_{i,\alpha} \otimes e_{j,\beta}) := \sigma(\alpha, \beta) \, s(e_{i,\alpha} \otimes e_{j,\beta})$.

For the odd color Hecke R-matrix (10) there is the following relation

$$(s \otimes' s) \circ R(\alpha, \beta) = (\bar{R}(\pi(\alpha), \pi(\beta))\hat{\otimes}R_\delta(\alpha, \beta))(s \otimes' s), \tag{24}$$

where $\bar{R}$ is a super-R-matrix on $\bar{E}$ (see [25, 26])

$$\bar{R} = \bigoplus_{a,b \in Z_2} ((-1)^a q^{1-2a} \Sigma_{i \in I_a} e_{i,a\ i,a} \otimes e_{i,a\ i,a} + \Sigma_{i \in I_a, j \in I_b, i \neq j} (-1)^{ab} e_{j,b\ i,a} \otimes e_{i,a\ j,b}$$

$$+ (q + q^{-1}) \Sigma_{i \in I_a, j \in I_b, i<j} e_{i,a\ i,a} \otimes e_{j,b\ j,b}), \tag{25}$$

R_δ is given on the space E' by the formula (5), $\delta i = \pi(\alpha)$, $\delta j = \pi(\beta)$, and the tensor product $\bar{R} \hat{\otimes} R_\delta$ is defined on $\tilde{E} \otimes E'$ by the following formula $\bar{R} \hat{\otimes} R' = P(\bar{R} \otimes R')P^{-1}$, where $P : (\tilde{E} \otimes E')^{\otimes 2} \longrightarrow \tilde{E}^{\otimes 2} \otimes E'^{\otimes 2}$, and $P((\tilde{e}_{i,\alpha} \otimes \rho_\alpha) \otimes (\tilde{e}_{j,\beta} \otimes \rho_\beta)) = (\tilde{e}_{i,\alpha} \otimes \tilde{e}_{j,\beta}) \otimes (\rho_\alpha \otimes \rho_\beta)$ (see [19]).

Note that if we have $\Gamma_0 = \Gamma$, then the color Hecke R-matrix (24) is even and reduces in a similar way to the ordinary R-matrix of the series A_{n-1}, [25].

3 DIFFERENTIAL CALCULI

Here we generalize the Wess-Zumino (see [27]) notion of noncommutative differential calculi on quantum plane to the graded case. First we generalize the concept of quantum (super-) plane to the graded case. Let E be a Γ-graded vector space equipped with a homogeneous basis $e_{i,\alpha}$, $i \in I_\alpha$, $\alpha \in \Gamma$. Let B be an arbitrary graded Yang-Baxter operator on E and let $\{B(\alpha, \beta) : \alpha, \beta \in \Gamma\}$ be a corresponding set of exchange operators.

An algebra defined as a quotient space $\mathcal{A} \equiv \mathcal{A}(E) := TE/I_B$, where I_B is an ideal generated by elements

$$\begin{aligned} &e_{i,\alpha} \otimes e_{j,\beta} - B_{i,\alpha\ j,\beta}^{k,\beta\ l,\alpha}\, e_{j,\beta} \otimes e_{l,\alpha}, && \text{for}\ \ i \neq j, \\ &e_{i,\alpha} \otimes e_{i,\alpha},\ \ \text{if}\ \ \varepsilon_\alpha = -1, && \text{for}\ \ i = j \end{aligned} \tag{26}$$

is said to be *a color (or holomorphic) quantum super-plane* for the odd gradation and *a color plane* for the even one.

The above definition means that $\mathcal{A}$ is an algebra generated by $x_{i,\alpha}$, $i \in I_\alpha$ and relations

$$\begin{aligned} &x_{i,\alpha}\, x_{j,\beta} = B_{i,\alpha\ j,b}^{k,\beta\ l,\alpha}\, x_{k,\beta}\, x_{l,\alpha}, && \text{for}\ \ i \neq j, \\ &(x_{i,\alpha})^2 = 0,\ \ \text{if}\ \ \varepsilon_\alpha = -1, && \text{for}\ \ i = j, \end{aligned} \tag{27}$$

where $x_{i,\alpha} := P(e_{i,\alpha})$, $P : TE \longrightarrow \mathcal{A}$ is the corresponding projection.

For the differential d on a color quantum (super-) plane $\mathcal{A}$ we assume the following standard properties

$$d = dx_{i,\alpha} \partial_{i,\alpha},\ \ d^2 = 0,\ \ d(uv) = (du)v + vdu, \tag{28}$$

10

where $u, v \in E$. We assume that we have here the following exchange relations

$$x_{i,\alpha} \, dx_{j,\beta} = C^{k,\beta \; l,\alpha}_{i,\alpha \; j,\beta} dx_{k,\beta} \, x_{l,\alpha}, \;\; \partial_{i,\alpha} \, \partial_{j,\beta} = F^{k,\beta \; l,\alpha}_{i,\alpha \; j,\beta} \partial_{k,\beta} \, \partial_{l,\alpha}, \tag{29}$$

and

$$\partial_{i,\alpha} \, x_{j,\beta} = 1 + (C^{-1})^{k,\beta \; l,\alpha}_{i,\alpha \; j,\beta} x_{k,\beta} \, \partial_{l,\alpha}. \tag{30}$$

for the noncommutative differentials and derivatives, respectively, C and F are certain graded Yang-Baxter operators. We can calculate that there are the following consistency conditions

$$
\begin{aligned}
&\left[(E_{12} - B_{12}) \circ (E_{12} + C_{12}) \right] (\alpha, \beta) = 0, \\
&\left[(E_{12} + C_{12}) \circ (E_{12} - F_{12}) \right] (\alpha, \beta) = 0, \\
&B_{12}(\beta, \gamma) B_{23}(\alpha, \gamma) B_{12}(\alpha, \beta) = B_{23}(\alpha, \beta) B_{12}(\alpha, \gamma) B_{23}(\beta, \gamma), \\
&C_{12}(\beta, \gamma) C_{23}(\alpha, \gamma) C_{12}(\beta, \gamma) = C_{23}(\alpha, \beta) C_{12}(\alpha, g) C_{23}(\beta, \gamma), \\
&B_{12}(\beta, \gamma) C_{23}(\alpha, \gamma) C_{12}(\alpha, \beta) = C_{23}(\alpha, \beta) C_{12}(\alpha, \gamma) B_{23}(\beta, g), \\
&C_{12}(\beta, \gamma) C_{23}(\alpha, \gamma) F_{12}(\alpha, \beta) = F_{23}(\alpha, \beta) C_{12}(\alpha, \gamma) C_{23}(\beta, \gamma).
\end{aligned}
\tag{31}
$$

for the set of exchange operators corresponding to B, C, and F. The above consistency conditions are generalization of those of Wess and Zumino [27] for the calculi on quantum plane.

The solution corresponding to the color Hecke braiding (11) is given by the relations

$$
\begin{aligned}
B(\alpha, \beta) &= F(\alpha, \beta) = q^{-1+2\pi(\alpha)} R(\alpha, \beta), \\
C(\alpha, \beta) &= q^{1-2\pi(\alpha)} R(\alpha, \beta).
\end{aligned}
\tag{32}
$$

We can see that this solution defines a consistent differential calculi on the quantum space in the Γ-graded case. If we substitute the solution (32) into formulae (27), (29) and (30), then we obtain

$$
\begin{aligned}
&x_{i,\alpha} \, x_{j,\beta} = q^{-1+2\pi(\alpha)} \varepsilon(\alpha, \beta) \, x_{j,\beta} \, x_{i,\alpha}, \; \text{for} \;\; i \neq j, \\
&(x_{i,\alpha})^2 = 0, \;\; \text{if} \;\; \varepsilon_{i,\alpha} = -1, \;\; \text{and} \;\; i = j, \\
&\partial_{i,\alpha} \, x_{i,\alpha} = 1 + q^{2(1-\pi(\alpha))} \, x_{i,\alpha} \, \partial_{i,\alpha} + \Sigma_{k>i}(q^{2(1-\pi(\alpha))} - 1) \, x_{k,\gamma} \, \partial_k, \\
&\partial_{i,\alpha} \, x_{j,\beta} = 1 + q^{-1+2\pi(\alpha)} \, x_{j,\beta} \, \partial_{i,\alpha}.
\end{aligned}
\tag{33}
$$

Next one can consider the quantum groups corresponding to the color Hecke braiding as a new example of braided groups of Majid [19]. One can also see that the differential calculus considered here is covariant with respect to these quantum groups.

4 CONCLUSIONS

Next one can generalize the deformations of the universal enveloping algebras of the Lie algebras A_{n-1} corresponding to the color Hecke braiding. The multiparameter deformations of the universal enveloping algebras introduced in [36] can be considered as a particular example of such generalization corresponding to the arbitrary gradation.

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